ON TWO CONSEQUENCES OF CH ESTABLISHED BY SIERPIŃSKI. II

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Abstract. We continue a study of the relations between two consequences of the Continuum Hypothesis discovered by Wacław Sierpiński, concerning uniform continuity of continuous functions and uniform convergence of sequences of real-valued functions, defined on subsets of the real line of cardinality continuum.

1. Introduction

In [11] we studied the following two consequences of the Continuum Hypothesis (CH) distinguished by Wacław Sierpiński in his classical treaty *Hypothèse du continu* [14] (the notation is taken from [14]):

C_8 There exists a continuous function $f : E \to \mathbb{R}$, $E \subseteq \mathbb{R}$, $|E| = \mathfrak{c}$, not uniformly continuous on any uncountable subset of E.

C_9 There is a sequence of functions $f_n : E \to \mathbb{R}$, $E \subseteq \mathbb{R}$, $|E| = \mathfrak{c}$, converging pointwise but not converging uniformly on any uncountable subset of E.

Sierpiński [13] checked that C_8 implies C_9. The status of the converse implication remains unclear. Let us notice, however, that in *Topology I* by Kuratowski [7], footnote (3) on page 533 suggests that the two statements are in fact equivalent.

In [11] we considered the following stratifications of statements C_8 and C_9 for uncountable cardinals $\kappa \leq \lambda \leq \mathfrak{c}$:

$C_8(\lambda, \kappa)$ There exists a set $E \subseteq \mathbb{R}$ of cardinality λ and a continuous function $f : E \to \mathbb{R}$, which is not uniformly continuous on any subset of E of cardinality κ.

$C_9(\lambda, \kappa)$ There exists a set $E \subseteq \mathbb{R}$ of cardinality λ (equivalently: for any set $E \subseteq \mathbb{R}$ of cardinality λ) and there is a sequence of functions $f_n : E \to \mathbb{R}$, converging on E pointwise but not converging uniformly on any subset of E of cardinality κ.

In particular, we proved in [11] that:

- $C_8(\mathfrak{c}, \mathfrak{c}) \Leftrightarrow C_9(\mathfrak{c}, \mathfrak{c})$, and each of these statements is equivalent to the assertion $\mathfrak{d} = \mathfrak{c}$, provided that the cardinal $\mathfrak{c}$ is regular.

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- $C_8(\aleph_1, \aleph_1) \Leftrightarrow C_9(\aleph_1, \aleph_1)$, and each of these statements is equivalent to the assertion $\mathfrak{b} = \aleph_1$.

Here $\mathfrak{d}$ and $\mathfrak{b}$ denote, as usual, the smallest cardinality of a dominating and, respectively, an unbounded family in $\mathbb{N}^{\mathbb{N}}$ corresponding to the ordering of eventual domination $\leq^*$ (cf. [4]).

An important role in our considerations was played by the notion of a K-Lusin set (cf. [1]) which we extended declaring that an uncountable subset E of a Polish space X is a *κ-K-Lusin set in X*, $\aleph_1 \leq \kappa \leq \mathfrak{c}$, if $|E \cap K| < \kappa$ for every compact set $K \subseteq X$. We proved in [11] that $C_9(\lambda, \kappa)$ is equivalent to the statement that there is a Polish space X and a κ-K-Lusin set of cardinality λ in X.

In this note we present additional results related to the subject of [11]. Most of them were earlier announced in [11, Section 4].

In Sections 2, 3 and 4 we investigate C_8-like phenomena in a more general setting. We are interested in two closely related problems (X and Y are fixed separable metric spaces).

Problem 1. Is the existence of a set $E \subseteq X$ of cardinality λ and a continuous function on E with values in Y, which is not uniformly continuous on any subset of E of cardinality κ, related to either $C_8(\lambda, \kappa)$ or $C_9(\lambda, \kappa)$?

Problem 2. Does the existence of a set $E \subseteq X$ of cardinality λ and a continuous function on E with values in Y, which is not uniformly continuous on any subset of E of cardinality κ, imply that there also exists such a function on E with values in $\mathbb{R}$?

Concerning Problem 1, we observe that the existence of a separable metric space X of cardinality λ, a metric space Y, and a continuous function on X with range in Y, which is not uniformly continuous on any subset of X of cardinality κ, already implies that there exists a κ-K-Lusin set of cardinality λ in some Polish space and, consequently, that $C_9(\lambda, \kappa)$ holds true (cf. Proposition 2.1).

Conversely, $C_9(\lambda, \kappa)$ implies that there exists a κ-K-Lusin set E of cardinality λ in $\mathbb{P}$, the set of irrationals of the unit interval $I = [0, 1]$, such that for every non-σ-compact Polish space Y there is a continuous function on E which is not uniformly continuous on any subset of E of cardinality κ (cf. Theorem 2.2).

On the other hand, if Y is compact, then the existence of a set $E \subseteq I$ of cardinality λ, and a continuous function $f : E \to Y$, which is not uniformly continuous on any subset of E of cardinality κ, implies $C_8(\lambda, \kappa)$ (cf. Theorem 3.2).

Concerning Problem 2, we show that if a set E in the Hilbert cube $I^{\mathbb{N}}$ is zero-dimensional and there exists a continuous function on E with range in any uncountable compact metric space Y, not uniformly continuous on any subset of X of cardinality κ, then there is also such a function with range in any uncountable compact metric space Z, and in particular, in I (cf. Theorem 3.1).

On the other hand, assuming CH, we prove the existence of a set $E \subseteq I^{\mathbb{N}}$ of cardinality $\mathfrak{c}$ such that there is a continuous function $f : E \to I^{\mathbb{N}}$, which is not uniformly continuous on any subset of E of cardinality $\mathfrak{c}$ but each continuous function $g : E \to \mathbb{R}$ is constant on a subset of E of cardinality $\mathfrak{c}$. The construction of a witnessing pair E and f falls under a general scheme, described in Section 4, of constructions of C_8-like examples based on a generalization of the notion of a κ-K-Lusin set.

Section 5 is a slight departure from the topic but it is closely related to a reasoning of Sierpiński concerning C_9. We shall show that a Hausdorff space X is Čech-complete and Lindelöf if and only if there is a sequence $f_0 \geq f_1 \geq \ldots$ of continuous functions $f_n : X \to I$ converging pointwise to zero but not converging uniformly on any closed non-compact set in X (cf. Theorem 5.2). The existence of such a sequence for any Polish space X was a key step in proving that statement $C_9(\lambda, \kappa)$ is equivalent to the existence of a κ-K-Lusin set E of cardinality λ in $\mathbb{P}$ (cf. [11, Theorem 2.3]).

In this note $\mathbb{P}$ always denotes the set of irrationals of the unit interval $I = [0, 1]$. It is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$, the countable product of the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ with the discrete topology (cf. [6]).

2. Mappings into non-compact spaces and C_9

We start with a general observation.

Proposition 2.1. *If there exist a separable metric space (X, d_X) of cardinality λ, a metric space (Y, d_Y), and a continuous function on X with range in Y, which is not uniformly continuous on any subset of X of cardinality κ, then there exists a κ-K-Lusin set of cardinality λ in some Polish space and, consequently, $C_9(\lambda, \kappa)$ holds true.*

Proof. Let $f : X \to Y$ be a continuous function which is not uniformly continuous on any subset of X of cardinality κ. The function f extends (cf. [6, Theorem 3.8]) to a continuous function $\tilde{f} : G \to \hat{Y}$ over a G_δ-set G (with $X \subseteq G$) in the Polish completion $(\hat{X}, \hat{d_X})$ of (X, d_X) into the completion $(\hat{Y}, \hat{d_Y})$ of (Y, d_Y). Now, if $K \subseteq G$ is compact, then $\tilde{f}$ is uniformly continuous on K, hence f is uniformly continuous on $K \cap X$. By the choice of X, we have that $|K \cap X| < \kappa$, which shows that X is a κ-K-Lusin set of cardinality λ in G. By [11, Theorem 2.3], this proves $C_9(\lambda, \kappa)$.

□

On the other hand, statement $C_9(\lambda,\kappa)$ already implies (and in view of Proposition 2.1, is equivalent to) the existence of C_9-like example for functions with values in arbitrary non-σ-compact Polish spaces.

Theorem 2.2. *For any uncountable cardinals $\kappa \le \lambda \le \mathfrak{c}$ the following are equivalent:*

(1) $C_9(\lambda,\kappa)$*,*

(2) there exist a set $E \subseteq \mathbb{R}$ of cardinality λ, a non-σ-compact Polish space Y, and a continuous function $f : E \to Y$ which is not uniformly continuous (with respect to arbitrary complete metric on Y) on any subset of E of cardinality κ.

Moreover, any κ-K-Lusin set E of cardinality λ in $\mathbb{P}$ has the property expressed in (2) with respect to any non-σ-compact Polish space Y.

Proof. (1) $\Rightarrow$ (2). By Theorem [11, Theorem 2.3], statement $C_9(\lambda,\kappa)$ implies that there is a κ-K-Lusin set E of cardinality λ in $\mathbb{P}$.

Let Y be an arbitrary non-σ-compact Polish space. Let $h : \mathbb{P} \to Y$ be a homeomorphic embedding of $\mathbb{P}$ onto a closed subspace $h(\mathbb{P})$ of Y (cf. [6, Theorem 7.10]). We will show that E together with $f = h|E$ have the required properties.

To that end, let us fix a set $A \subseteq E$ with $|A| = \kappa$. Then $\bar{A}$, the closure of A in I, is not contained in $\mathbb{P}$ since otherwise $\bar{A}$ would be a compact set in $\mathbb{P}$, intersecting E on a set of cardinality κ. So let us pick $a_k \in A$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} a_k = a$ and $a \in I \setminus \mathbb{P}$.

Now, if f were uniformly continuous on A with respect to a complete metric d on Y, f would take Cauchy sequences in A to Cauchy sequences in Y. In particular, the sequence $(f(a_n))_{n\in\mathbb{N}}$ would be Cauchy in Y, hence $\lim_{k\to\infty} f(a_k) = z$ for some $z \in Y$. This, however, is not the case: since the set $\{a_0, a_1, \dots\}$ has no accumulation point in $\mathbb{P}$, the set $\{f(a_0), f(a_1), \dots\}$ has no accumulation point in $h(\mathbb{P})$ and hence also in Y, as $h(\mathbb{P})$ is closed in Y.

The implication (2) $\Rightarrow$ (1) follows immediately from Proposition 2.1. □

3. Mappings into compact spaces and C_8

The results of the previous section show that C_8-like statements for functions with values in non-σ-compact Polish spaces are actually equivalent to statement C_9. Apparently, the situation changes when we consider functions with values in compact spaces. As the following result shows, if a set $E \subseteq I^{\mathbb{N}}$ is zero-dimensional and there exists a continuous function on E with range in any uncountable compact metric space Y, not uniformly continuous on any subset of X of cardinality κ, then there is also such a function with values in $\mathbb{R}$, witnessing that $C_8(\lambda,\kappa)$ holds true.

Theorem 3.1. *Let E be a zero-dimensional subset of a compact metric space X. If there exists a continuous function on E with range in a compact metric space Y, not uniformly continuous on any subset of E of cardinality κ, then there is also such a function with range in the Cantor ternary set C in I. Consequently, for any uncountable compact metric space Z, there is also such a function with values in Z.*

Proof. Let $h : E \to Y$ be a continuous function not uniformly continuous on any set of cardinality κ.

Using the compactness of Y, let as fix a sequence $(K_n, L_n)_{n\in\mathbb{N}}$ of pairs of disjoint compact sets in Y such that for any pair (K, L) of disjoint compact sets in Y, there is n with $K \subseteq K_n$ and $L \subseteq L_n$.

For each $n \in \mathbb{N}$, we let $C_n = h^{-1}(K_n)$, $D_n = h^{-1}(L_n)$, and using the fact that E is zero-dimensional and separable, we choose a continuous function $u_n : E \to \{0, 2\}$ taking on C_n value 0 and on D_n value 2, cf. [3, Theorem 1.2.6].

We shall show that the function $f : E \to I$ defined by the formula

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{3^{i+1}} u_i(x),$$

which takes values in the Cantor ternary set C, is not uniformly continuous on any subset of E of cardinality κ.

To that end, let us fix a set $A \subseteq E$ with $|A| = \kappa$.

We shall first make the following observation. Let $a \in \bar{A}$, the closure of A in X. Then for any n, since u_n takes on A values 0 or 2 only, the oscillation

$$\inf\{\operatorname{diam}(u_n(A \cap V)) : V \text{ is a neighbourhood of } a \text{ in } X\}$$

of $u_n|A$ at a is either 0 or 2.

Let us note that $h|A : A \to Y$ cannot be extended to a continuous (hence, by the compactness of $\bar{A}$, uniformly continuous) function $\bar{h} : \bar{A} \to Y$, since otherwise the function $h|A = \bar{h}|A$ would itself be uniformly continuous, contrary to the the fact that $|A| = \kappa$. Consequently, there must be closed disjoint sets K, L in Y such that the closures of $(h|A)^{-1}(K)$ and $(h|A)^{-1}(L)$ in X meet, cf. [2, Theorem 3.2.1]. Then for an n with $K \subseteq K_n$ and $L \subseteq L_n$, we infer that $\overline{C_n \cap A} \cap \overline{D_n \cap A} \neq \emptyset$, so let us fix $a \in \overline{C_n \cap A} \cap \overline{D_n \cap A}$.

It follows that $u_n|A$ has the oscillation at a equal to 2 and let us assume that n is the smallest index with this property. The oscillation of each of the functions $u_0|A, \ldots, u_{n-1}|A$ is then equal to 0, hence we can find a neighbourhood V of a in X such that all these functions have constant values on $A \cap V$.

Let us pick $x_k,\ y_k \in A\cap V$, $k\in\mathbb{N}$, such that $\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = a$ and for every k, $u_n(x_k)=0$, $u_n(y_k)=2$, hence

$$f(y_k)-f(x_k)=\tfrac{2}{3^{n+1}}+\sum_{i=n+1}^{\infty}\tfrac{1}{3^{i+1}}(u_i(y_k)-u_i(x_k)) \quad \geq$$
$$\tfrac{2}{3^{n+1}}-\sum_{i=n+1}^{\infty}\tfrac{2}{3^{i+1}}=\tfrac{1}{3^{n+1}}.$$

It follows that the oscillation of $f|A$ at a is at least 3^{-n-1}. In effect, $f|A$ has no continuous extension over $\bar{A}$, which means that f is not uniformly continuous on A.

Finally, if (Z,d) is an arbitrary compact metric space and $e: C\to Z$ is a homeomorphic embedding of C into Z, then since e^{-1} is uniformly continuous, the function $e\circ f: E\to Z$ has desired properties.

□

As an immediate corollary we obtain the following equivalent form of statement $C_8(\lambda,\kappa)$.

Theorem 3.2. *For any uncountable cardinals $\kappa\le\lambda\le\mathfrak{c}$ if the cofinality of λ is uncountable, then the following are equivalent:*

(1) $C_8(\lambda,\kappa)$,

(2) there exists a set $E\subseteq\mathbb{R}$ of cardinality λ, a compact metric space Y and a continuous function $f: E\to Y$, which is not uniformly continuous on any subset of E of cardinality κ.

Moreover, any set $E\subseteq I$ that witnesses $C_8(\lambda,\kappa)$ for some continuous function from E to I has the property expressed in (2) with respect to any uncountable compact metric space.

Proof. (1) ⇒ (2). If $f: E\to\mathbb{R}$ is a continuous function on a set $E\subseteq\mathbb{R}$ of cardinality λ, which is not uniformly continuous on any subset of E of cardinality κ, then since the cofinality of λ is uncountable, by shrinking E, if necessary, we may assume that the range of f is contained in a closed interval Y of length 1.

If additionally $E\subseteq I$ and $f: E\to I$, then since E contains no non-trivial interval, it is zero-dimensional, and Theorem 3.1 applies.

(2) ⇒ (1). Now let $f: E\to Y$ be a continuous function with values in a compact metric space Y, which is not uniformly continuous on any subset of E of cardinality κ. We may again assume that E is contained in a closed interval X. Then, E being zero-dimensional, it is enough to apply the final part of the assertion of Theorem 3.1 to $Z=I$.

□

By the results of [11] (cf. Section 1), Theorems 2.2 and 3.2 lead to the following corollary.

Corollary 3.3. *If either $\kappa=\lambda=\mathfrak{c}$ or $\kappa=\lambda=\aleph_1$, then the following are equivalent:*

(1) *There exist a set $E \subseteq \mathbb{R}$ of cardinality λ, a non-σ-compact Polish space Y, and a continuous function $f : E \to Y$ which is not uniformly continuous (with respect to any complete metric on Y) on any subset of E of cardinality κ.*
(2) *There exist a set $E \subseteq \mathbb{R}$ of cardinality λ, a compact metric space Y and a continuous function $f : E \to Y$, which is not uniformly continuous on any subset of E of cardinality κ.*

4. Constructing C_8-like examples from $\mathscr{K}$-Lusin sets

Throughout this section we assume that G is an uncountable G_δ-set in a compact metric space X.

Given a collection $\mathscr{K}$ of compact sets in X containing all singletons, we say that an uncountable subset E of G is a *κ-$\mathscr{K}$-Lusin set in G*, where $\aleph_1 \leq \kappa \leq \mathfrak{c}$, if $|E \cap K| < \kappa$ for every set $K \in \mathscr{K}$. In particular, if $\mathscr{K} = K(G)$, the collection of all compact sets in G, then a κ-$\mathscr{K}$-Lusin set in G is just a κ-K-Lusin set in G.

The C_8-like examples, presented later in this section by means of $\mathscr{K}$-Lusin sets, are based on the following observation, applied also in the proofs of [11, Theorems 3.8 and 3.9].

Proposition 4.1. *Let $\varphi : X \to Y$ be a continuous map onto a compact metric space Y such that $\varphi|G$ is a homeomorphism onto $\varphi(G)$. Let $\mathscr{K}$ be a collection of compact sets in X such that whenever $A \subseteq G$ and $\varphi|A$ extends to a homeomorphism over $\bar{A}$, the closure of A in X, then $\bar{A} \in \mathscr{K}$.*

If $T \subseteq G$ is a κ-$\mathscr{K}$-Lusin set in G of cardinality λ, where $\aleph_1 \leq \kappa \leq \lambda \leq \mathfrak{c}$, then letting $E = \varphi(T)$ and $f = \varphi^{-1}|E : E \to X$, we obtain a continuous function on a set of cardinality λ, which is not uniformly continuous on any subset of E of cardinality κ.

Proof. Aiming at a contradiction, assume that $f|B$ is uniformly continuous (with respect to any metric compatible with the topology of Y) on a set $B \subseteq E$ of cardinality κ and let $A = f(B) = \varphi^{-1}(B)$. Then, since $\varphi|A : A \to B$ is also uniformly continuous, the function $\varphi|A$ extends to a homeomorphism over $\bar{A}$ (cf. [2, Theorem 4.3.17]). Consequently, $\bar{A} \in \mathscr{K}$. This, however, is impossible, since on one hand we have $|T \cap \bar{A}| < \kappa$, T being a κ-$\mathscr{K}$-Lusin set in X, but on the other hand $A \subseteq T \cap \bar{A}$ has cardinality κ. □

4.1. **Zero-dimensional spaces.**
Throughout this subsection let us additionally assume that the (compact metric) space X is zero-dimensional.

A proof of the following fact is given in [9, Lemma 4.2] (it is based on an idea similar to that in [3, proof of Lemma 5.3]).

Lemma 4.2. *For any G_δ-set G in X there is a continuous map $\varphi : X \to Y$ onto a compact metric space Y, such that $\varphi|G$ is a homeomorphism onto $\varphi(G)$, $\varphi(X \setminus G) \cap \varphi(G) = \emptyset$ and the set $Y \setminus \varphi(G)$ is countable.*

Various C_8-like examples could be constructed with the help of Lemma 4.2 and the following observation. By a *σ-ideal on X* we mean a collection $\mathcal{I}$ of Borel sets in X, closed under taking Borel subsets and countable unions of elements of $\mathcal{I}$, and containing all singletons.

Proposition 4.3. *Let $\mathcal{I}$ be a σ-ideal on X and let $\mathscr{K}$ be the collection of all compact sets from $\mathcal{I}$.*

If G is a G_δ set in X such that $G \in \mathcal{I}$ and $\varphi : X \to Y$ is a continuous map described in Lemma 4.2, then $\bar{A} \in \mathscr{K}$ for any $A \subseteq G$ such that $\varphi|A$ extends to a homeomorphism onto $\bar{A}$.

Consequently, if $T \subseteq G$ is a κ-$\mathscr{K}$-Lusin set in G of cardinality λ, where $\aleph_1 \leq \kappa \leq \lambda \leq \mathfrak{c}$, then letting $E = \varphi(T)$ and $f = \varphi^{-1}|E : E \to X$, we obtain a continuous function on a set of cardinality λ, which is not uniformly continuous on any subset of E of cardinality κ.

Proof. Let us fix $A \subseteq G$ such that $\varphi|A$ extends to a homeomorphism $\tilde{\varphi}$ between $\bar{A}$ and $\overline{\varphi(A)}$. Then the set $\bar{A} \setminus G = \tilde{\varphi}^{-1}\big(\overline{\varphi(A)} \setminus \varphi(G)\big)$ is countable, so $\bar{A} \subseteq G \cup (\bar{A} \setminus G) \in \mathcal{I}$, hence $\bar{A} \in \mathscr{K}$.

The final assertion follows directly from Proposition 4.1.

□

As the following proposition shows, the above observation could be applied to various natural σ-ideals. Let us recall that un uncountable set T in a Polish space Y is a *Lusin set in Y*, if $|T \cap D| < \aleph_1$ for every closed nowhere-dense subset of Y.

Proposition 4.4. *Let $\mathcal{I}$ be a σ-ideal on X, let $\mathscr{K}$ be the collection of all compact sets from $\mathcal{I}$ and let us assume that $\mathcal{I}$ is not generated by $\mathscr{K}$ (i.e., there is a set from I which is not covered by any F_σ-set from $\mathcal{I}$). Then, there exists a G_δ-set in X such that $G \in \mathcal{I}$ but no non-empty, relatively open set in G is covered by an F_σ-set from $\mathcal{I}$. Consequently, every Lusin set T in G is $\aleph_1$-$\mathscr{K}$-Lusin in G hence, it gives rise to a C_8-example (as described in Proposition 4.3).*

Proof. Let $B \in \mathcal{I}$ be (a Borel) set not covered by any F_σ-set from $\mathcal{I}$. Then the existence of a G_δ-set in X such that $G \subseteq B$ but G is not covered by any F_σ-set from $\mathcal{I}$ follows from a theorem of Solecki (see [15]). By shrinking G, if necessary, we may assume that G has the desired properties.

It follows that if $K \in \mathscr{K}$, then $G \cap K$ is meager in G, and, consequently, $T \cap K$ is countable for any Lusin set T in G. □

Remark 4.5. *A typical example of the situation described in Propositions 4.3 and 4.4 is when X is a copy of the Cantor in $\mathbb{R}$ of positive*

Lebesgue measure, $\mathcal{I}$ is the σ-ideal of Lebesgue measure zero Borel sets in X (then $\mathscr{K}$ is the family of closed Lebesgue measure zero sets in X) and G is a dense copy of irrationals in X of Lebesgue measure zero.

Then the function $f = \varphi^{-1}|H : H \to \mathbb{R}$, where $H = \varphi(G)$, is a homeomorphic embedding of H, a copy of irrationals, to $\mathbb{R}$, with the property that for every Lusin set L of cardinality $\mathfrak{c}$ in H, the function $f|L$ is not uniformly continuous on any uncountable subset of L. This provides an alternative proof of the theorem of Sierpiński that the existence of a Lusin set of cardinality $\mathfrak{c}$ in $\mathbb{P}$ implies C_8 (cf. [14, proof of Théorème 6 on page 45]*).*

4.2. **Infinite-dimensional spaces.** The assumption that the space X is zero-dimensional in Theorem 3.1 is essential, as demonstrated by the following result (CH in this theorem can be weakened to the assumption that no family of less than $\mathfrak{c}$ meager sets covers $\mathbb{R}$, cf. Remark 4.8).

Theorem 4.6. *Assuming CH, there exists a set $E \subseteq I^{\mathbb{N}}$ of cardinality $\mathfrak{c}$ such that*

(1) there is a continuous function $f : E \to I^{\mathbb{N}}$, which is not uniformly continuous on any subset of E of cardinality $\mathfrak{c}$,
(2) each continuous function $g : E \to \mathbb{R}$ is constant on a subset of E of cardinality $\mathfrak{c}$.

A key element of the proof of this theorem is a *Henderson compactum* – a compact metrizable infinite-dimensional space whose all compact subsets of finite dimension are zero-dimensional, cf. [10].

More specifically, we shall need the following fact, where *punctiform* sets are the sets without non-trivial subcontinua, cf. [3, 1.4.3].

Lemma 4.7. *There exists a non-empty punctiform G_δ-set M in a Henderson compactum $H \subseteq I^{\mathbb{N}}$ such that for each $M' \subseteq M$ with $\dim\,(M \setminus M') \leq 0$, every continuous function $u : M' \to \mathbb{R}$ is constant on a set of positive dimension.*

A justification of this lemma is rather standard, but since we did not find convenient references, we shall give a proof to this effect at the end of this subsection. For now, taking this fact for granted, we shall proceed with a proof of our main result.

Proof of Theorem 4.6. Let us adopt the notation of Lemma 4.7, and let $\mathscr{K}$ be the collection of all compact zero-dimensional sets in the Henderson compactum H.

Using CH, we inductively construct a $\mathfrak{c}$-$\mathscr{K}$-Lusin set T in M such that $|T \cap L| = \mathfrak{c}$ for every Borel set L in M with $\dim L > 0$. To that end, we list all elements of $\mathscr{K}$ as $\langle K_\alpha : \alpha < \mathfrak{c}\rangle$ and all Borel sets L in M with $\dim L > 0$ as $\langle L_\alpha : \alpha < \mathfrak{c}\rangle$, repeating each such set L continuum many times. Then, we subsequently pick for $\xi < \mathfrak{c}$

$$p_\xi \in L_\xi \setminus \big(\bigcup_{\alpha<\xi} K_\alpha \cup \{p_\alpha : \alpha < \xi\}\big),$$

using the fact that, under CH, the set in brackets is zero-dimensional, by the sum theorem for dimension zero (cf. [3, 1.3.1]). Finally, we let

$$T = \{p_\xi : \xi < \mathfrak{c}\}.$$

In particular, T meets each Borel set in M of positive dimension in continuum many points.

Claim 1. Each continuous function $w : T \to \mathbb{R}$ is constant on a set of cardinality $\mathfrak{c}$.

Indeed, w can be extended to a continuous function $\tilde{w} : \tilde{T} \to \mathbb{R}$ over a G_δ-set $\tilde{T}$ in M containing T. The Borel set $M \setminus \tilde{T}$ is disjoint from T, hence dim $(M \setminus \tilde{T}) \leq 0$. Setting in Lemma 4.7 $u = \tilde{w}$ and $M' = \tilde{T}$, we get $r \in \mathbb{R}$ such that dim $u^{-1}(r) > 0$. Being a Borel set in M, $u^{-1}(r)$ intersects T in a set of cardinality $\mathfrak{c}$, i.e., $|w^{-1}(r)| = \mathfrak{c}$.

Let us now apply a counterpart of Lemma 4.2 for infinite-dimensional spaces to the following effect (cf. the proof of [3, Lemma 5.3.1]): there exists a continuous surjection $\varphi : H \to Y$ onto a compact metrizable space Y such that $\varphi|M$ is a homeomorphism onto $\varphi(M)$, $\varphi(H \setminus M) \cap \varphi(M) = \emptyset$ and $Y \setminus \varphi(M)$ is a countable union of finite-dimensional compact sets.

Claim 2. The set $E = \varphi(T)$ satisfies the assertion of Theorem 4.6.

Since E is homeomorphic to T, Claim 1 shows that (2) in Theorem 4.6 is satisfied.

To check also assertion (1) in this theorem, we shall make sure that the function

$$f = \varphi^{-1}|E : E \to H$$

is not uniformly continuous on any set of cardinality $\mathfrak{c}$.

Since T is a $\mathfrak{c}$-$\mathscr{K}$-Lusin set in M, by Proposition 4.1 it is enough to verify that for any non-empty $A \subseteq M$, whenever $\varphi|\bar{A}$ is an embedding (the closure of A is in H), then $\bar{A} \in \mathscr{K}$, i.e., dim $\bar{A} = 0$.

To check this, let us note that

$$\varphi(\bar{A} \setminus M) = \varphi(\bar{A}) \setminus \varphi(M) \subseteq Y \setminus \varphi(M)$$

is a countable union of compact finite-dimensional sets, and so is $\bar{A} \setminus M$. Since H is a Henderson compactum, $\bar{A} \setminus M$ is a countable union of zero-dimensional compact sets, hence $\dim(\bar{A} \setminus M) = 0$ by the sum theorem (cf. [3, 1.4.5]). By the enlargement theorem, cf. [3, 1.5.11], $\bar{A} \setminus M$ can be enlarged to a zero-dimensional G_δ-set L in H. The set $\bar{A} \setminus L$ is a σ-compact subset in the punctiform space M, hence again by the sum theorem, dim $(\bar{A} \setminus L) \leq 0$ (cf. [3, 1.4.5]). By the addition theorem, cf. [3, 1.5.10], dim $\bar{A} \leq 1$, and H being a Henderson compactum, dim $\bar{A} = 0$.

□

In the assumptions of Theorem 4.6, CH can be weakened to the assertion (usually denoted by $\mathrm{cov}(\mathscr{M}) = \mathfrak{c}$, cf. [5]) that no family of less than $\mathfrak{c}$ meager sets covers $\mathbb{R}$. The only change in the proof requires checking that the inductive definition of the sequence $\langle p_\xi : \xi < \mathfrak{c}\rangle$ is correct. More precisely, the following is true.

Remark 4.8. *Assuming* $\mathrm{cov}(\mathscr{M}) = \mathfrak{c}$*, the union of any family* $\mathscr{K}$ *of compact zero-dimensional sets in* $I^{\mathbb{N}}$*, with* $|\mathscr{K}| < \mathfrak{c}$*, is zero-dimensional.*

Proof. Let $\mathscr{K}$ be a family of compact zero-dimensional sets in $I^{\mathbb{N}}$, with $|\mathscr{K}| < \mathfrak{c}$. For any $K \in \mathscr{K}$, let us consider

(1) $\mathscr{U}_K = \{f \in C(I^{\mathbb{N}}, \mathbb{R}) : \; f^{-1}(0) \cap K = \emptyset\}$.

Then $\mathscr{U}_K$ is open and dense in the function space $C(I^{\mathbb{N}}, \mathbb{R})$ and this space being perfect Polish, the assumption $\mathrm{cov}(\mathscr{M}) = \mathfrak{c}$ guarantees that the set

(2) $\mathscr{G} = \bigcap\{\mathscr{U}_K : K \in \mathscr{K}\}$

is dense in $C(I^{\mathbb{N}}, \mathbb{R})$ (cf. [6, 8.32]).

Let us recall (cf. [3]) that a closed set L in a topological space Z *separates the space* Z *between sets* $A_1,\ A_2 \subseteq Z$, if $Z \setminus L = U_0 \cup U_1$, where U_0, U_1 are open, disjoint and $A_i \subseteq U_i$, for $i = 0, 1$. If A_1 and A_2 are singletons, then we say that L separates Z between the respective points.

Let $F = \bigcup \mathscr{K}$. To prove that dim $F = 0$, it is enough to show that for any pair of disjoint compact sets $A,\ B$ in $I^{\mathbb{N}}$, $I^{\mathbb{N}}$ can be separated between A and B by a closed set disjoint from F. Indeed, let $x \in F$ and let U be a relatively open subset of F with $x \in U$ and $F \setminus U \neq \emptyset$. Then $U = V \cap F$ for an open V in $I^{\mathbb{N}}$ with $B = I^{\mathbb{N}} \setminus V \neq \emptyset$, and separating $I^{\mathbb{N}}$ between $A = \{x\}$ and B by a closed set in $I^{\mathbb{N}}$, disjoint from F, leads to disjoint open sets V_1, V_2 in $I^{\mathbb{N}}$ with $x \in V_1 \cap F \subseteq U$, $V_2 \cap F \neq \emptyset$, and $F = (V_1 \cap F) \cup (V_2 \cap F)$. Thus, $V_1 \cap F$ is a relatively clopen in F neighbourhood of x contained in U.

So let $A,\ B$ be disjoint compact sets in $I^{\mathbb{N}}$. We can pick $f \in \mathscr{G}$ such that $f(A) \subseteq (-\infty, 0)$, $f(B) \subseteq (0, +\infty)$, as such functions form an open, non-empty set in $C(I^{\mathbb{N}}, \mathbb{R})$. Then $L = f^{-1}(0)$ is a closed set in $I^{\mathbb{N}}$ separating $I^{\mathbb{N}}$ between A and B with the property that $L \cap K = \emptyset$ for all $K \in \mathscr{K}$, cf. (1) and (2).

□

Proof of Lemma 4.7. Let us fix a Henderson continuum K in $I^{\mathbb{N}}$. We shall consider on $I^{\mathbb{N}}$ the metric assigning to points $(s_0, s_1, \ldots)$, $(t_0, t_1, \ldots)$ the distance $\sum_i 2^{-i}|s_i - t_i|$.

Claim 1. There exists a punctiform G_δ-set S in K and distinct points $a, b \in S$ such that each relatively closed set in S separating S between a and b has dimension at least 2.

Indeed, since $\dim K \geq 4$, K contains a punctiform G_δ-set W with $\dim W \geq 3$.

This theorem goes back to Mazurkiewicz, and can be justified by the reasoning in the proof of [10, Theorem 3.9.3] applied to a pair of disjoint compact sets A and B in K such that for any set N in K with $\dim N \leq 2$ there is a continuum in K joining A and B and missing N (cf. [12, Theorem 4.2]), and to a continuous map $\pi : K \to [-1, 1]$ sending A to -1 and B to 1.

Now, since $\dim W \geq 3$, there is $a \in W$ such that all sufficiently small neighbourhoods of a in W have boundaries of dimension ≥ 2. An argument in [8, proof of Theorem 8, p. 172] gives a point $b \in K \setminus \{a\}$ such that for $S = W \cup \{b\}$, every relatively closed set in S separating S between a and b has dimension ≥ 2. Since S is a punctiform G_δ-set in K, it satisfies Claim 1.

Claim 2. There are embeddings $h_n : I^{\mathbb{N}} \to I^{\mathbb{N}}$, $n = 1, 2, \ldots$, such that, letting

$$M_0 = S \times \{0\},\ M_n = h_n(S) \times \{\tfrac{1}{n}\} \text{ for } n = 1, 2, \ldots,$$
$$M = \bigcup_{n=0}^{\infty} M_n \subseteq I^{\mathbb{N}} \times I \text{ and } H = \overline{M} \text{ – the closure of } M \text{ in } I^{\mathbb{N}} \times I,$$

one obtains sets $M \subseteq H$ satisfying the assertion of Lemma 4.7.

We adopt the notation from Claim 1. Since K is a compact, connected set in $I^{\mathbb{N}}$, we can find open connected neighbourhoods $U_1 \supseteq U_2 \supseteq \ldots$ of K such that each open neighbourhood of K in $I^{\mathbb{N}}$ contains some U_n.

Let D be a countable set dense in S, and let $\langle (c_n, d_n) : n \in \mathbb{N} \rangle$ be an enumeration of all ordered pairs of distinct points from D, such that each such pair appears in the sequence infinitely many times.

Let us fix $n > 0$. We shall define an embedding $h_n : I^{\mathbb{N}} \to U_n$ such that the distance from $h_n(a)$ to c_n and the distance from $h_n(b)$ to d_n is less than $\frac{1}{n}$.

To that end, let us notice that U_n is arcwise connected, being an open, connected set in $I^{\mathbb{N}}$ (cf. [16, Proposition 12.25]), and hence there is a continuous $f : I^{\mathbb{N}} \to U_n$ with $f(a) = c_n$, $f(b) = d_n$. Let $m \geq n$ be large enough to ensure that for any $x, y \in I^{\mathbb{N}}$, whenever the first m coordinates of x and $f(y)$ coincide, then $x \in U_n$. Now, denoting by $p_j : I^{\mathbb{N}} \to I$ the projection onto j'th coordinate, we define the embedding h_n by $p_i(h_n(x)) = p_i(f(x))$ for $i \leq m$ and $p_{m+i}(h_n(x)) = p_i(x)$ for $i = 1, 2, \ldots$.

Having defined the embeddings h_n, we shall check that the set M in Claim 2 satisfies the assertion of Lemma 4.7.

Clearly, M is a punctiform G_δ-set in $I^{\mathbb{N}} \times I$.

Since each neighbourhood of K in $I^{\mathbb{N}}$ contains all but finitely many $h_n(K) \subseteq \overline{U_n}$, the set

$$K \times \{0\} \cup \bigcup_{n=1}^{\infty} \Big(h_n(K) \times \Big\{ \frac{1}{n} \Big\} \Big)$$

contains $H = \overline{M}$, and since it is clearly a Henderson compactum containing M, so is H.

Let $M' \subseteq M$ satisfy dim $(M \setminus M') \leq 0$ and let $u : M' \to \mathbb{R}$ be continuous.

Since dim $(M_0) =$ dim $(S) \geq 2$, the addition theorem yields that dim $(M' \cap M_0) \geq 1$. Therefore, if u is constant on $M' \cap M_0$, we are done.

Let us assume that this is not the case. Then for some $x, y \in M' \cap M_0$, $u(x) \neq u(y)$, hence for $r = \frac{u(x)+u(y)}{2}$, $u^{-1}(r)$ is a relatively closed set in M' which separates M' between x and y. One can extend $u^{-1}(r)$ to a closed set L in H such that L separates H between x and y, cf. [7, §21, XI, Theorem 2, p. 226]. Let us pick $c, d \in D$, close enough to x and y, respectively, so that L separates H between $(c, 0)$ and $(d, 0)$, and using the fact that $(c, d) = (c_n, d_n)$ for infinitely many n's and the distances from $h_n(a)$ to c_n, and from $h_n(b)$ to d_n tend to zero, let us pick n such that L also separates H between $(h_n(a), \frac{1}{n})$ and $(h_n(b), \frac{1}{n})$. In effect, since $x \mapsto (h_n(x), \frac{1}{n})$ is a homeomorphism of S onto M_n, and each relatively closed set separating S between a and b has dimension ≥ 2, we infer that dim $(M_n \cap L) \geq 2$. Since dim $(M_n \setminus M') \leq 0$, applying again the addition theorem, we conclude that dim $(M' \cap L) \geq 1$.

Finally, since $M' \cap L = u^{-1}(r)$, u is constant on a set of positive dimension.

□

5. A characterization of complete metrizability

A key step in proving that statement $C_9(\lambda, \kappa)$ is equivalent to the existence of a κ-K-Lusin set E of cardinality λ in $\mathbb{P}$ (cf. [11, Theorem 2.3]) is Theorem 2.1 in [11] (closely related to a theorem of Sierpiński, cf. [11, Remark 2.2]) stating that for any Polish space X there is a sequence $f_0 \geq f_1 \ldots$ of continuous functions $f_n : X \to I$ which converges to zero pointwise but does not converge uniformly on any set with non-compact closure in X.

In fact, the existence of such a function sequence characterizes complete metrizability of a separable metrizable space X.

Theorem 5.1. *A separable metrizable space X is completely metrizable if and only if there is a sequence $f_0 \geq f_1 \geq \ldots$ of continuous functions $f_n : X \to I$ which converges to zero pointwise but does not converge uniformly on any set with non-compact closure in X.*

Proof. In view of [11, Theorem 2.1] we only have to prove the "if" part of the above equivalence. So let (X,d) be a separable metric space. Let $f_n : X \to I$, $n \in \mathbb{N}$, be a sequence of continuous functions with $f_0 \geq f_1 \geq \ldots$, which converges to zero pointwise but does not converge uniformly on any set with non-compact closure in X.

Let $(\hat{X}, \hat{d})$ be the completion of (X,d); clearly, the space $\hat{X}$ is Polish. There exists a G_δ-set $\tilde{X}$ in $\hat{X}$, containing X such that each f_n extends to a continuous function $\tilde{f}_n : \tilde{X} \to I$. Since X is dense in $\tilde{X}$, we have $\tilde{f}_0 \geq \tilde{f}_1 \geq \ldots$ and in particular, the set

$$G = \{x \in \tilde{X} : \lim_{n\to\infty} \tilde{f}_n(x) = 0\}$$

is G_δ in $\tilde{X}$.

Since $X \subseteq G$, it remains to make sure that $G \subseteq X$, to conclude that $X = G$ is completely metrizable as a G_δ-subset of the (Polish) space $\hat{X}$.

So let $c \in G$ and let us pick $x_k \in X$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} x_k = c$. Let $K = \{c\} \cup \{x_k : k \in \mathbb{N}\}$. Then K is compact, so the sequence $(\tilde{f}_n)_{n\in\mathbb{N}}$ converges uniformly on K (this is a special instance of the Dini theorem [2, Lemma 3.2.18]). It follows that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $K \cap X$, a closed set in X which therefore, by the assumed property of $(f_n)_{n\in\mathbb{N}}$, is compact. Since the sequence $(x_n)_{n\in\mathbb{N}}$ converges to c and all x_n are elements of $K \cap X$, so is c. In particular, $c \in X$.

□

With the help of Theorem 5.1 we shall establish the following more general result (for terminology see [2]).

Theorem 5.2. *Let X be a Hausdorff space. Then X is a Čech-complete Lindelöf space if and only if there is a sequence $f_0 \geq f_1 \geq \ldots$ of continuous functions $f_n : X \to I$ converging pointwise to zero but not converging uniformly on any closed non-compact set in X.*

Proof. First, let X be a Čech-complete Lindelöf space. By a theorem of Frolík, it follows that there is a perfect map $p : X \to Y$ onto a Polish space Y, cf. [2, 5.5.9]. Let us recall that for a Hausdorff space X and a metrizable space Y this means, cf. [2, Theorems 3.7.2 and 3.7.18], that p is a continuous, closed mapping and the inverse image of every compact subset of Y is compact. It is straightforward to check that if functions $f_n : Y \to I$, $n \in \mathbb{N}$, with $f_0 \geq f_1 \geq \ldots$, satisfy the assertion of Theorem 5.1, the functions $f_n \circ p : X \to I$, $n \in \mathbb{N}$, have the required properties.

Next, given a sequence $f_0 \geq f_1 \geq \ldots$ of functions $f_n : X \to I$, described in the theorem, let us consider the diagonal map

(1) $F = (f_0, f_1, \ldots) : X \to I^{\mathbb{N}}$ and let $Y = F(X)$.

Let $p_n : I^{\mathbb{N}} \to I$ denote the projection onto the n'th coordinate and let, cf. (1), for $n \in \mathbb{N}$,

(2) $g_n = p_n|Y : Y \to I$.

Note that, since $f_n = g_n \circ F$, the properties of the sequence $(f_n)_{n\in\mathbb{N}}$ are passed to the sequence $(g_n)_{n\in\mathbb{N}}$, namely:

(3) $g_0 \geq g_1 \geq \ldots$ is a sequence of continuous functions converging pointwise to zero,

and for any closed set A in Y,

(4) if $(g_n)_{n\in\mathbb{N}}$ converges on A uniformly, then A is compact.

To see (4), let us consider $B = F^{-1}(A)$. Then, as $f_n = g_n \circ F$, $(f_n)_{n\in\mathbb{N}}$ converges on B uniformly, hence B is compact, and so is $A = F(B)$.

By Theorem 5.1,

(5) Y is a G_δ-set in $I^{\mathbb{N}}$.

To complete the proof, it is enough to show that

(6) the mapping F is perfect.

Indeed, if (6) holds, then since by (5), Y is a Čech-complete Lindelöf space, so is its perfect preimage X, cf. [2, Theorems 3.8.9 and 3.9.10].

To prove (6), it suffices to check that for every compact $K \subseteq Y$ the inverse image $F^{-1}(K)$ is compact, cf. [2, Theorem 3.7.18].

So let $K \subseteq Y$ be compact. Then, again by the Dini theorem, the sequence $(g_n)_{n\in\mathbb{N}}$ converges uniformly on K, and since $f_n = g_n \circ F$, cf. (2), the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $F^{-1}(K)$. By the assumed property of this sequence, $F^{-1}(K)$ is compact, completing the proof.

□

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